Integrable generalizations of the two-dimensional Born–Infeld equation

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Abstract. The Born–Infeld equation in two dimensions is generalized to higher dimensions. Lorentz invariance is retained, and the resulting system is completely integrable via linearization by Legendre transformation. This generalization retains homogeneity in second derivatives of the field.

1. Introduction

There are many nonlinear integrable equations now known in more than two dimensions; the Davy–Stewartson equation [1], the Kadomtsev–Petviashvilli equation [2] and the Konopelchenko–Rogers equation [3] being well known examples. Most of such equations, including these particular ones, suffer from a lack of Lorentz invariance, so are inappropriate as field theoretic models with particle-like solutions. In the case of $1+1$ dimensions, there are two well known integrable nonlinear equations which are Lorentz invariant, which admit localized solutions and have been used to model fundamental particles; the Sine–Gordon equation [4] and the Born–Infeld equation [5], which in light-cone coordinates in $1+1$ dimensions is given by

$$\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial t^2} + \left(\frac{\partial\phi}{\partial t}\right)^2\frac{\partial^2\phi}{\partial x^2} - \left(\lambda + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right)\frac{\partial^2\phi}{\partial x\partial t} = 0. \tag{1.1}$$

The parameter λ, if non-zero, can be scaled to unity. This equation is known to be integrable [6–8]. In this article we should like to view it as an integrable deformation of the Bateman equation, which corresponds to (1.1) when $\lambda = 0$, in analogy with the integrable deformations of conformal field theories proposed by Zamolodchikov [9]. The properties of covariance (if $\phi(x, t)$ is a solution to the Bateman equation, so is any function of $\phi(x, t)$) and of the existence of an infinite class of inequivalent Lagrangian densities from which the equation may be derived [10] are lost, but that of integrability is retained. Also the curious property of the Bateman equation that it is form invariant under arbitrary linear transformations of the coordinates (x, t), which means that the Bateman equation is 'signature blind', is replaced by Poincaré invariance.

In some recent work [10, 11, 13], one of us and his collaborators have proposed a class of completely integrable models which we call universal field equations, of which the Bateman equation is the two-dimensional prototype. This equation, describing a theory in d dimensions, may be written in the following equivalent forms

$$\det\begin{pmatrix} 0 & \dfrac{\partial\phi}{\partial x_j} \\ \dfrac{\partial\phi}{\partial x_i} & \dfrac{\partial^2\phi}{\partial x_i\partial x_j} \end{pmatrix} = 0 \tag{1.2}$$

or

$$\mathrm{Tr}(GA) = 0 \tag{1.3}$$

where the matrix G has components

$$G_{ij} = \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tag{1.4}$$

and

$$A_{ij} = \mathrm{Adj}\left(\frac{\partial^2 \phi}{\partial x_k \partial x_l}\right)_{ij} = \det\left(\frac{\partial^2 \phi}{\partial x_k \partial x_l}\right)\left(\frac{\partial^2 \phi}{\partial x_k \partial x_l}\right)^{-1}_{ij} \tag{1.5}$$

is the adjugate matrix of $M_{ij} = \partial^2 \phi / \partial x_i \partial x_j$. Under a Lorentz transformation generated by the matrix Λ, both G and A transform in the same way as

$$G' = \Lambda^{-1} G \Lambda \qquad A' = \Lambda^{-1} A \Lambda. \tag{1.6}$$

From (1.3) it is easy to see that the universal equation is invariant under Lorentz transformations. How may it be deformed in such a way as to retain the properties of Lorentz invariance and integrability? G is idempotent and so the only other matrix available to us which transforms according to (1.6) is the metric tensor η_{ij}. Thus the only candidate with the properties we require is the equation

$$\sum_{i,j} A_{ij}(G_{ji} + f(\mathrm{Tr}(\eta G))\eta_{ji}) = 0 \tag{1.7}$$

where f is an arbitrary function of the quadratic Lorentz invariant constructed from $\partial \phi / \partial x_i$. We are excluding the possibility of an explicit dependence upon ϕ, and retain the homogeneity in second derivatives. In the following considerations we shall take f to be a constant, λ, generalizing (1.1), with a brief mention of the more general case when we discuss linearization. Using the results of [13], we shall show how both of these forms give rise to equations linearizable by a Legendre transform. It is crucial for the proof of integrability given below that the second-order derivatives enter only through components of the adjugate matrix which appear linearly. The only other possibility, tractable by this method, is for an additional linear dependence upon $\det M$. Other Lorentz invariant equations could be constructed using powers of the matrix M, including the Barbashov–Chernikov generalization of the two-dimensional Born–Infeld equation to higher dimensions [7], but there is no incontrovertible evidence for the integrability of such equations, despite some hopeful remarks in the literature [14]. Note that neither this generalization nor the generalization presented here are equivalent to the original Born–Infeld equation [5], but the terminology seems to be established for the two-dimensional situation. The three-dimensional version of these equations is a candidate for an alternative theory of strings, as it describes the motion of a surface $\phi(x_1, x_2, x_3)$, the world sheet of a string. The case when $\lambda = 0$, which just describes developable surfaces [15] was treated in [11]. The characteristic property of such surfaces is that they contain straight lines.

   We are thus led to essentially three forms of nonlinear field equations in four dimensions which are Lorentz invariant and are integrable; the well known system of self-dual Yang–Mills equations and its supersymmetric extensions [16, 17] , the relativistic string equations

[18, 19], which are a bit of a cheat as the base space is two-dimensional, and the equations proposed here, which are directly related to linear equations through the Legendre transform.

The paper is organized as follows. In the next section a new derivation of some well known properties of the Born–Infeld equation in two dimensions is given. The proof of integrability goes back at least to [7] and is discussed further in [8, 20], but we shall present a slightly different version for completeness. Section 3 briefly reviews the Legendre transform method, and shows how the Lorentz invariant deformations of the universal field equation given by (1.1) may be solved implicitly. After a brief discussion of the $(2+1)$-dimensional case, the Lagrangian for the equation (1.7) is constructed. Unlike the case of the universal field equation, it is unique, thus resolving an ambiguity as to how such theories might be quantized by the Feynman path integral method. On the other hand, the equation of motion also follows from the same iterative procedure described in [10].

2. The Born–Infeld equation

The Born–Infeld equation in light cone coordinates in $1+1$ dimension is given by

$$\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial t^2} + \left(\frac{\partial\phi}{\partial t}\right)^2\frac{\partial^2\phi}{\partial x^2} - \left(\lambda + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right)\frac{\partial^2\phi}{\partial x\partial t} = 0 \qquad (2.1)$$

the parameter λ, if non-zero can be scaled to unity. This equation can be viewed as an integrable deformation of the Bateman equation, which corresponds to $\lambda = 0$, in analogy with the integrable deformations of conformal field theories proposed by Zamolodchikov [9].

This equation can be written as a first-order equation in a similar manner to the Bateman equation with the help of the two independent roots u_1, u_2 of the quadratic equation for the characteristics [6]

$$\left(\frac{\partial\phi}{\partial x}\right)^2 u^2 - \left(\lambda + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t}\right)u + \left(\frac{\partial\phi}{\partial t}\right)^2 = 0. \qquad (2.2)$$

The roots of this equation are

$$u_{\substack{1\\2}} = \frac{\lambda + 2\dfrac{\partial\phi}{\partial x}\dfrac{\partial\phi}{\partial t} \pm \sqrt{\lambda^2 + 4\lambda\dfrac{\partial\phi}{\partial x}\dfrac{\partial\phi}{\partial t}}}{2\left(\dfrac{\partial\phi}{\partial x}\right)^2}. \qquad (2.3)$$

The Born–Infeld equation can then be written in either of two forms

$$\frac{\partial u_1}{\partial t} = u_2\frac{\partial u_1}{\partial x} \qquad (2.4)$$

$$\frac{\partial u_2}{\partial t} = u_1\frac{\partial u_2}{\partial x}. \qquad (2.5)$$

These equations possess an infinite number of conservation laws; it is easy to verify that

$$\frac{\partial}{\partial t}(u_1 + u_2) = \frac{\partial}{\partial x}(u_1 u_2) \qquad (2.6)$$

$$\frac{\partial}{\partial t}(u_1^2 + u_1 u_2 + u_2^2) = \frac{\partial}{\partial x}(u_1 u_2(u_1 + u_2)) \qquad (2.7)$$

$$\frac{\partial}{\partial t}(u_1^3 + u_1^2 u_2 + u_1 u_2^2 + u_2^3) = \frac{\partial}{\partial x}(u_1 u_2(u_1^2 + u_1 u_2 + u_2^2)) \qquad (2.8)$$

In fact, if S_n denotes the symmetric polynomial of nth degree in u_1, u_2, then the general conservation law is

$$\frac{\partial}{\partial t} S_n = \frac{\partial}{\partial x} (u_1 u_2 S_{n-1}) \tag{2.9}$$

this is easily proved using $S_n = u_1^n + u_1 S_{n-1}$ and induction. The general solution of the equations for u_1, u_2 is an implicit one; the roles of dependent and independent variables may be interchanged to give

$$\frac{\partial x}{\partial u_2} = -u_2 \frac{\partial t}{\partial u_2}$$
$$\tag{2.10}$$
$$\frac{\partial x}{\partial u_1} = -u_1 \frac{\partial t}{\partial u_1}$$

with the solution

$$x = f(u_1) - u_1 f'(u_1) + g(u_2) - u_2 g'(u_2)$$
$$\tag{2.11}$$
$$t = f'(u_1) + g'(u_2)$$

where f, g are arbitrary functions and a prime denotes differentiation with respect to the argument. Note that this is still some way off a solution for ϕ, which requires a solution of the above equations for $\partial \phi / \partial x$, $\partial \phi / \partial t$ which may then in principle be integrated. There is a nice class of explicit solutions

$$\phi = F(\alpha x + \beta t) + \gamma x + \delta t \tag{2.12}$$

where F is arbitrary, and the constants α, β, γ, δ satisfy the polynomial equation

$$\lambda \alpha \beta + 2\alpha \beta \gamma \delta - \beta^2 \gamma^2 - \alpha^2 \delta^2 = 0. \tag{2.13}$$

The complete solution may be obtained in principle by the inversion of the equations for x, t in terms of u_1, u_2, and the subsequent integration of the equations

$$\frac{\partial \phi}{\partial x} = \frac{1}{\sqrt{u_1} - \sqrt{u_2}} \qquad \frac{\partial \phi}{\partial t} = \frac{\sqrt{u_1 u_2}}{\sqrt{u_1} - \sqrt{u_2}}. \tag{2.14}$$

The consistency condition which guarantees integrability of these equations is simply the Born–Infeld equation itself. Thus in principle this equation is as fully integrable as is the Bateman equation, even though, according to the analysis of [21], it admits only a single Lagrangian, namely $\{\lambda + 4(\partial \phi / \partial x)(\partial \phi / \partial t)\}^{1/2}$.

Here is an example of a solution generated by the results (2.11) and (2.14). A particularly convenient choice of f and g in the system (2.11) is $f(u_1) = u_1^2$ and $g(u_2) = -u_2^2$. This gives explicit expressions for u_1 and u_2 in terms of x and t

$$u_1 = -(4x + t^2)/4t$$
$$\tag{2.15}$$
$$u_2 = -(4x - t^2)/4t.$$

These can be substituted back into the formulae (2.14) which can then be integrated to give an explicit solution

$$\phi = -\frac{\sqrt{2}}{3} \left(\left(\frac{4x - t^2}{-2t} \right)^{3/2} + \left(\frac{4x + t^2}{-2t} \right)^{3/2} \right) + \text{constant}. \tag{2.16}$$

3. Legendre transforms

The Legendre transform, which was used in [13] to linearize the universal field equation may also be used to linearize (1.7). This transform, which is clearly involutive, has the flavour of a twistor transform.

The multivariable version of this transform runs as follows [22]. Introduce a dual space with coordinates ξ_i, $i = 1, \ldots, d$ and a function $w(\xi_i)$ defined by

$$\phi(x_1, x_2, \ldots, x_d) + w(\xi_1, \xi_2, \ldots, \xi_d) = x_1\xi_1 + x_2\xi_2 + \ldots, x_d\xi_d \tag{3.1}$$

$$\xi_i = \frac{\partial\phi}{\partial x_i} \qquad x_i = \frac{\partial w}{\partial \xi_i} \qquad \forall i. \tag{3.2}$$

To evaluate the second derivatives ϕ_{ij} in terms of derivatives of w it is convenient to introduce two Hessian matrices, Φ and W with matrix elements ϕ_{ij} and $w_{\xi_i\xi_j} = w_{ij}$, respectively. Then assuming that Φ is invertible, $\Phi W = 1$ and

$$\frac{\partial^2\phi}{\partial x_i\partial x_j} = (W^{-1})_{ij} \qquad \frac{\partial^2 w}{\partial\xi_i\partial\xi_j} = (\Phi^{-1})_{ij}. \tag{3.3}$$

The effect of the Legendre transformation upon the equation (1.7) is immediate; in the new variables the equation becomes simply

$$\sum_{i,j}\left(\xi_i\xi_j + f\left(\sum\xi_k^2\right)\eta_{ji}\right)\frac{\partial^2 w}{\partial\xi_i\partial\xi_j} = 0 \tag{3.4}$$

a linear second-order equation for w. All sums are implicitly taken with the Lorentzian metric. Introducing the variable $\rho = \left(\sum\xi_k^2\right)^{1/2}$ this equation takes the form

$$\left[(\rho^2 + f(\rho))\frac{\partial^2}{\partial\rho^2} + \frac{df(\rho) - 1}{\rho}\frac{\partial}{\partial\rho} + \frac{f(\rho)}{2\rho^2}\sum_{i,j}\left(\xi_i\frac{\partial}{\partial\xi_j} - \xi_j\frac{\partial}{\partial\xi_i}\right)^2\right]w = 0. \tag{3.5}$$

Single-valued solutions of this equation are easily obtained when it is realized that the eigenfunctions of the generalized total angular momentum operator $\sum_{i<j}(\xi_i(\partial/\partial\xi_j) - \xi_j(\partial/\partial\xi_i))^2$ are just harmonic functions on the $d-1$ sphere, with eigenvalues $-n(n+d-2)$, n integral. Then the general solution can be found by the method of separation of variables as $w = \sum_n F_n(\rho) \times$ (general harmonic of degree n), where $F_n(\rho)$ is a solution to the ordinary differential equation

$$(\rho^2 + f(\rho))\frac{d^2 F_n}{d\rho^2} + \frac{df(\rho) - 1}{\rho}\frac{dF_n}{d\rho} - n(n + d - 2)\frac{f(\rho)}{\rho^2}F_n = 0. \tag{3.6}$$

Given such a solution, a parametric representation for x_i in terms of ξ_j can be constructed from $x_i = \partial w/\partial\xi_i$, and these relations, together with the definition of w in terms of ϕ, are sufficient to eliminate the variables ξ_j and solve for ϕ.

Of course this procedure is only a solution in principle; in practice there will be comparatively few solutions for w for which an explicit solution can be obtained.

4. The case of $2 + 1$ dimensions

In $2 + 1$ dimensions our equation takes the form

$$\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\phi_{i\alpha}\phi_{j\beta}(\phi_\gamma\phi_k - f(\rho)\eta_{\gamma k}) = 0. \tag{4.1}$$

Here repeated indices are summed, ϕ_j denotes $\partial\phi/\partial x_j$ and ρ is defined as in the previous section. Even the simplest case in $(2 + 1)$ dimensions, where the arbitrary function $f(\rho)$ is just a constant, λ, is awkward. A more convenient choice is $f(\rho) = -\rho^2$. Then, the ODE (3.6) becomes

$$\frac{3\rho^2 + 1}{\rho}\frac{dF_n}{d\rho} - n(n+1)F_n = 0. \tag{4.2}$$

This provides the solution for F_n

$$F_n = F_{0_n}(3\rho^2 + 1)^{n(n+1)/6} \tag{4.3}$$

which permits a general solution for ω. No special case has been found to be invertible in closed form for $\phi(x_1, x_2, x_3)$.

The construction of an infinite set of conservation laws for the equation (4.1) has also defied analysis as yet. However we can demonstrate a form for the universal equation, analogous to the Bateman case discussed in [10] as follows. Define $u = \phi_2/\phi_1$, $v = \phi_3/\phi_1$ then (4.1) in the case where $f = 0$ becomes

$$\begin{vmatrix} u_2 - uu_1 & u_3 - vu_1 \\ v_2 - uv_1 & v_3 - vv_1 \end{vmatrix} = 0 \tag{4.4}$$

in a form suitable for generalization to d dimensions, or else, more explicitly as

$$u_2v_3 - vu_3v_2 - u(u_1v_3 - v_1u_3) - v(u_2v_1 - u_1v_2) = 0. \tag{4.5}$$

A subsidiary condition is that the matrix in (4.4) is symmetric.

There are an infinite number of conservation laws of the form

$$\frac{\partial}{\partial x_1}\left[\frac{u^m}{m}\frac{\partial}{\partial x_2}\left(\frac{v^{n+1}}{n+1}\right) - \left(\frac{v^{n+1}}{n+1}\right)\frac{\partial}{\partial x_2}\left(\frac{u^m}{m}\right) + \frac{v^n}{n}\frac{\partial}{\partial x_3}\left(\frac{u^{m+1}}{m+1}\right)\right.$$
$$\left. - \left(\frac{u^{m+1}}{m+1}\right)\frac{\partial}{\partial x_3}\left(\frac{v^n}{n}\right)\right]$$
$$+ \frac{\partial}{\partial x_2}\left[\left(\frac{u^m}{m}\right)\frac{\partial}{\partial x_3}\left(\frac{v^n}{n}\right) - \left(\frac{v^n}{n}\right)\frac{\partial}{\partial x_3}\left(\frac{u^m}{m}\right) - \left(\frac{u^m}{m}\right)\frac{\partial}{\partial x_1}\left(\frac{v^{n+1}}{n+1}\right)\right.$$
$$\left. + \left(\frac{v^{n+1}}{n+1}\right)\frac{\partial}{\partial x_1}\left(\frac{u^m}{m}\right)\right]$$
$$+ \frac{\partial}{\partial x_3}\left[\left(\frac{v^n}{n}\right)\frac{\partial}{\partial x_2}\left(\frac{u^m}{m}\right) - \left(\frac{u^m}{m}\right)\frac{\partial}{\partial x_2}\left(\frac{v^n}{n}\right) - \left(\frac{v^n}{n}\right)\frac{\partial}{\partial x_1}\left(\frac{u^{m+1}}{m+1}\right)\right.$$
$$\left. + \left(\frac{u^{m+1}}{m+1}\right)\frac{\partial}{\partial x_1}\left(\frac{v^n}{n}\right)\right] = 0. \tag{4.6}$$

It seems reasonable to expect that this procedure should generalize to the deformed equation (4.1), but we have not been able to accomplish this.

5. Lagrangian derivation

In contrast to the universal field equation, which possesses an infinite number of inequivalent Lagrangian functions of which it is the resultant variation, the modification (1.7) admits only one [21]. However this Lagrangian retains one feature of the fully covariant situation; it may be expressed in terms of an iteration of Euler variations. Denote by $\mathcal{E}$ the Euler differential operator

$$\mathcal{E} = -\frac{\partial}{\partial \phi} + \partial_i \frac{\partial}{\partial \phi_{x_i}} - \partial_i \partial_j \frac{\partial}{\partial \phi_{x_i,x_j}} \cdots \tag{5.1}$$

(in principle the expansion continues indefinitely but it is sufficient here to terminate at the stage involving the variational operator $\partial/\partial \phi_{x_i x_j}$). Now consider the Lagrangian density

$$L_1 = \left(\sum_{i,j} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \eta_{ij} + \lambda \right)^{1/2} \tag{5.2}$$

with equation of motion

$$\mathcal{E} L_1 = \frac{1}{L_1^3} \sum_{ij} \left(L_1^2 \eta_{ij} - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right) \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0 \tag{5.3}$$

where all the contractions are taken using the Lorentz metric. In the case where the dimension of spacetime is two, (5.3) is simply the two-dimensional Born–Infeld equation (2.1). Now relax the imposition of zero on the right-hand side of (5.3) and define a new Lagrangian density, $L_2 = L_1 \mathcal{E} L_1$, with equation of motion

$$\mathcal{E} L_2 = \frac{1}{L_1^3} \sum_{ij} \sum_{kl} \left(L_1^2 \eta_{ij} \eta_{kl} - \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_j} \eta_{il} \right)$$
$$\times \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial^2 \phi}{\partial x_k \partial x_l} - \frac{\partial^2 \phi}{\partial x_i \partial x_k} \frac{\partial^2 \phi}{\partial x_j \partial x_l} \right) = 0. \tag{5.4}$$

In the case of a three-dimensional spacetime this equation is just (1.7). The general pattern is clear; defining recursively the density L_n by $L_n = L_1 \mathcal{E} L_{n-1}$, the equation in d dimensions is simply given by $L_{d-1} = 0$. This is in exact parallel with the iterative generation of the universal field equation, where the only difference is that instead of a specific choice for L_1, any function of ϕ_{x_j} which is homogeneous of weight one, with a vanishing Hessian $\det(\phi_{x_j,x_k})$ will do [10].

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